THOMSON REUTERS ENDNOTE

[Kumar*, 5(2): February, 2016] ISSN: 2277-9655

(I2OR), Publication Impact Factor: 3.785

IJESRT

INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

THERMOSOLUTAL CONVECTION IN A HETEROGENEOUS VISCO-ELASTIC (OLDROYDIAN) FLUID LAYER IN A POROUS MEDIUM

Sudhir Kumar

Associate Professor, S.D. (PG) College, Muzaffarnagar (UP).

ABSTRACT

The paper critically examines, within the framework of linear stability analysis, the thermosolutal convection in a heterogeneous visco-elastic (Oldroydian) fluid layer in a porous medium. In the present paper, stationery, oscillatory and non-oscillatory convection have been discussed in details. A variational principle is established for the present problem. Some Results are discussed numerically also. Also, principle of exchange of stabilities is not valid in the present problems.

KEYWORDS: Thermosolutal instability, Heterogeneous fluid, Oldroydian fluid, Porous Medium.

INTRODUCTION

The problem of onset of convection in a horizontal layer of fluid heated from below, when buoyancy forces arise from density difference due to variation in temperature, was studied by Benard **[1]** and Rayleigh **[2]** and the problem, under varying assumptions of hydromagnetics, has been treated in detail by Chandrashekhar **[3]**, Investigations of thermosolutal convection, when buoyancy forces arise also from variations in solute concentration apart from those due to variation in temperature, are motivated by its direct relevance to the hydrodynamics of oceans, as well as its interesting complexities, as a double-diffusion phenomenon. Stomell et al. **[4]** did the pioneering work in this direction. Since then the problem of thermohaline or thermosolutal convection has been studied in three basic configurations by Stern **[5]**, Veronis **[6]** and Nield **[7]** respectively, when the fluid layer heated and soluted from above, heated and soluted from below and heated from below and soluted form above.

An experimental demonstration by Toms and Strabridge **[8]** has revealed that a dilute solution of methyl methacrylate in n-butyl acetate agrees well with the theoretical model of Oldroydian visco-elastic fluid proposed by Oldroyd **[9]**. Sharma **[10]** has studied the instability of the plane interface between two Oldroydian visco-elastic superposed conducting fluids in the presence of a uniform magnetic field.

In view of the fact that the study of visco-elastic fluid in a porous medium finds, applications in geophysics and chemical technology, a number of researchers have contributed in this direction. However, the thermosolutal convection in a heterogeneous visco-elastic (Oldroydian) fluid layer in a porous medium seems, to the best, of our knowledge, uninvestigated so far.

In this paper, therefore, we have examined the stability of a visco-elastic [Oldroydian] fluid layer heated and soluted from below in a porous medium first studied by Khare and Sahai [11] leading to an adverse temperature gradient and a solute concentration gradient with free boundaries when the initial non-homogeneity is present in the fluid. Hence it can be looked upon as an extension of thermosolutal convection in a homogenous fluid layer in porous medium discussed by Khare and Sahai.

CONSTITUTIVE EQUATIONS AND THE EQUATIONS OF MOTION

Let T_{ij}, $\square_{ij}$, e_{ij}, $\square_{ij}$, p, q_i, □, □, $\square_0$, T and C denote respectively the total stress tensor, shear stress tensor, rate of strain tensor, Kronecker delta, scalar pressure, velocity vector of the fluid, viscosity, stress relaxation time, strain retardation time, temperature field and concentration field. Then the Oldroydian visco-elastic fluid is described by the constitutive equations

ISSN: 2277-9655
(I2OR), Publication Impact Factor: 3.785

$$\left.\begin{aligned} T_{ij} &= -p\delta_{ij} + \tau_{ij} \\ \left[1+\lambda\frac{d}{dt}\right]\tau_{ij} &= 2\mu\left[1+\lambda_0\frac{d}{dt}\right]e_{ij} \\ \text{and} \qquad e_{ij} &= \frac{1}{2}\left[\frac{\partial q_i}{\partial x_j}+\frac{\partial q_j}{\partial x_i}\right] \end{aligned}\right] \tag{1}$$

where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{q}.\nabla$ is the total derivative being the sum of local and convective derivatives.

Let us consider a horizontal layer of saturated porous medium of thickness d between two free-boundaries z = 0 and z = d, z-axis being vertically upward. Let the interstitial fluid (fluid in the porous medium) be visco-elastic, incompressible and heterogeneous. The initial inhomogeneity in the fluid is assumed to be of the form □$_0$f(z), where f(z) is a monotonic function of z with f(0) = 1 and is such that $\frac{df}{dz}$ is constant. The layer is infinite in horizontal directions and is heated and soluted from below leading to an adverse temperature gradient $\beta = \frac{(T_0 - T_1)}{d}$ and a uniform solutal gradient $\beta' = \frac{(S_0 - S_1)}{d}$, where T_0 and T_1 are the constant temperatures of the lower and upper boundaries with $T_0 > T_1$ and also S_0 and S_1 are the constant solute concentrations of the lower and upper surfaces with $S_0 > S_1$. The effective density is the superposition of inhomogeneity described by

(a) □ = □$_0$f(z)

and (b) □ = □$_0$[1 + □(T_0 – T) – □′(S_0 – S)]

which is caused by temperature and solute gradients. This leads to the effective density

$$\rho = \rho_0[f(z) + \alpha(T_0 - T) - \alpha'(S_0 - S)]. \tag{2}$$

When the fluid flows through a porous medium, the gross effect is represented by Darcy's law. As a result, the usual viscous term is replaced by the resistance term $-\left[\frac{\mu}{k_1}\right]\mathbf{V}$, where k_1 and $\mathbf{V}$ denote respectively the medium permeability and the filter velocity.

Hence, the basic equations are

$$\rho_0\left[1+\lambda\frac{\partial}{\partial t}\right]\frac{D\mathbf{q}}{\partial t} = \left[1+\lambda\frac{\partial}{\partial t}\right][-\nabla p + \rho X_i] - \frac{\rho_0\nu}{k_1}\left[1-\lambda_0\frac{\partial}{\partial t}\right]\mathbf{q}, \tag{3}$$

$$\nabla.\mathbf{q} = 0 \tag{4}$$

$$\frac{\partial\rho}{\partial t} + (\mathbf{q}.\nabla)\rho = 0, \tag{5}$$

$$\frac{\partial T}{\partial t} + (\mathbf{q}.\nabla)T = k_T\nabla^2 T, \tag{6}$$

and
$$\frac{\partial C}{\partial t} + (\mathbf{q}.\nabla)C = k_S\nabla^2 C, \tag{7}$$

where $\mathbf{q}$ and $\nu = \left(= \dfrac{\mu}{\rho_0} \right)$ are respectively the velocity and kinematic viscosity of the fluid respectively, k_1 is the intrinsic permeability of the medium and $k_1 \to \infty$ corresponds to non-porous medium.

BASIC STATE AND THE PERTURBATION EQUATIONS

The initial stationary state, whose stability we wish to examine is that of an incompressible, viscous, visco-elastic (Oldroydian) fluid arranged in a horizontal strata in a non-homogeneous and isotropic porous medium. The system is acted upon by a temperature T, concentration C and the gravity field g (0, 0, –g).
The initial state whose stability we wish to examine is thus characterized by

$$\mathbf{q} = (0,0,0), T = T_0 - \beta z, C = C_0 - \beta' z, \rho = \rho_0[f(z) + \alpha\beta z - \alpha'\beta' z] \text{and } p = p_0 - \int_0^z g\rho dz. \tag{8}$$

where p_0 is the pressure when □ = □$_0$.
Following the usual procedure and normal mode technique given by

$$F(z) = \exp.[i(k_x x + k_y y) + nt], \tag{9}$$

where $k = \sqrt{k_x^2 + k_y^2}$ is the real wave number of propagation and n is the frequency of arbitrary disturbance.
We get

$$[1+FP_1\sigma]\left\{[\sigma(D^2-a^2)(D^2-a^2-\sigma P_1)][\tau(D^2-a^2)-\sigma P_1]w + \frac{R_2 a^2}{P_1\sigma}[D^2-a^2-\sigma P_1]\right.$$
$$[\tau(D^2-a^2)-\sigma P_1]w - R_1 a^2[\tau(D^2-a^2)-\sigma P_1]w + R'a^2(D^2-a^2-\sigma P_1)w$$
$$+P_2[1+F\varepsilon P_1\sigma][D^2-a^2-P_1\sigma][\tau(D^2-a^2)-\sigma P_1]w = 0. \tag{10}$$

where

$R = \dfrac{g\alpha\beta d^4}{k_T \nu}$ is the thermal Rayleigh number, $P_1 = \dfrac{\nu}{k_T}$ is the thermal Prandtl number,

$R' = \dfrac{g\alpha'\beta' d^4}{k_T \nu}$ is the concentration Rayleigh number, $\tau = \dfrac{k_S}{k_T}$ is the Lewis number,

$F = \dfrac{\lambda k_T}{d^2}$ is the elastic parameter, $R_2 = \dfrac{g d^4 \left(\dfrac{df}{dz}\right)}{k_T \nu}$, $P_2 = \dfrac{d^2}{k_1}$ and $\varepsilon = \dfrac{\lambda_0}{\lambda}$.

Equation (10) can also be written as

$$[1+FP_1\sigma][\sigma^2 P_1(D^2-a^2)(D^2-a^2-\sigma P_1)][\tau(D^2-a^2)-\sigma P_1]w + R_2 a^2[D^2-a^2-\sigma P_1]$$
$$[\tau(D^2-a^2)-\sigma P_1]w - \sigma P_1 R_1 a^2[\tau(D^2-a^2)-\sigma P_1]w + \sigma P_1 R' a^2(D^2-a^2-\sigma P_1)w$$
$$+\sigma P_1 P_2[1+F\varepsilon P_1\sigma][D^2-a^2-P_1\sigma][\tau(D^2-a^2)-\sigma P_1]w = 0. \tag{11}$$

The solution of the equation (11) is to be obtained under the following boundary conditions :

$w = D^2 w = 0 =$ □ = □ at z = 0 and z = 1. (12)

RESULTS AND DISCUSSION

(A) Stationary Convection

Let the marginal state be stationary, so that it is characterized by □ = 0.
Hence for stationary convection, equation (11) becomes

R^2a^2□$(D^2 - a^2)^2w = 0.$ (13)

Multiplying equation (13) by w* and integrating over the range of z, we have

$$R_2 a^2 \tau \int_0^1 (D^2 - a^2)^2 ww^* dz = 0$$

or

$$R_2 a^2 \tau \int_0^1 (|D^2 w|^2 + a^4 + 2a^2 |Dw|^2) dz = 0 \qquad (14)$$

In view of the boundary conditions (12) and following Chandrashekhar **[3]**, we find that w = 0, □ = 0, □ = 0 are the only possible solutions. Thus, we observe that the hypothesis that initial state solutions are perturbed is contradicted. Therefore, the instability can not set in as stationary convection, or in other words the Principle of Exchange of Stabilities (PES) is not valid for the problem under investigation.

(B) Oscillatory Convection

Now for the proper solution of equation (11) for w belonging to the lowest mode, we follow Chandrashekhar **[3]** and assume that solution w satisfying the boundary conditions is given by w = w_0 sin □z.
Equation (12) yields

$$[1+FP_1\sigma]\Big\{[\sigma^2 P_1(D^2-a^2)(D^2-a^2-\sigma P_1)][\tau(D^2-a^2)-\sigma P_1]w + R_2a^2[D^2-a^2-\sigma P_1]$$
$$[\tau(D^2-a^2)-\sigma P_1]w - \sigma P_1 R_1 a^2[\tau(D^2-a^2)-\sigma P_1]w + \sigma P_1 R' a^2(D^2-a^2-\sigma P_1)w$$
$$+\sigma P_1 P_2[1+F\varepsilon P_1\sigma](D^2-a^2)[D^2-a^2-P_1\sigma][\tau(D^2-a^2)-\sigma P_1]w = 0. \qquad (15)$$

Let $I_1 = [D^2-a^2][D^2-a^2-\sigma P_1][\tau(D^2-a^2)-\sigma P_1]w,\ I_2 = [D^2-a^2-\sigma P_1][\tau(D^2-a^2)-\sigma P_1]w,$

$I_3 = [\tau(D^2-a^2)-\sigma P_1]w$ and $I_4 = [D^2-a^2-\sigma P_1]w.$

Substituting w = w_0 sin □z in the expressions for I_1, I_2, I_3 and I_4, we get

$$I_1 = [D^2-a^2][D^2-a^2-\sigma P_1][\tau(D^2-a^2)-\sigma P_1]w_0 \sin\pi z$$
$$= -[\tau(\pi^2+a^2)+\sigma P_1][\pi^2+a^2+\sigma P_1][\pi^2+a^2]w_0 \sin\pi z \qquad (16)$$

$$I_2 = [D^2-a^2-\sigma P_1][\tau(D^2-a^2)-\sigma P_1]w_0 \sin\pi z,$$
$$= [\tau(\pi^2+a^2)+\sigma P_1][\pi^2+a^2+\sigma P_1]w_0 \sin\pi z, \qquad (17)$$

$$I_3 = [\tau(D^2-a^2)-\sigma P_1]w_0 \sin\pi z$$
$$= -[\tau(\pi^2+a^2)+\sigma P_1] \qquad (18)$$

and

$$I_4 = [D^2 w_0 \sin\pi z - a^2 w_0 \sin\pi z - \sigma P_1 w_0 \sin\pi z],$$
$$= -[\pi^2+a^2+\sigma P_1]w_0 \sin\pi z \qquad (19)$$

Substituting for I_1, I_2, I_3 and I_4 from (16) to (19) into equation (15), we have

$$\sigma^2 P_1[1+FP_1\sigma][\pi^2+a^2][\pi^2+a^2+\sigma P_1][\tau(\pi^2+a^2)+\sigma P_1] = R_2a^2[1+FP_1\sigma][\pi^2+a^2+\sigma P_1]$$

$$[\tau(\pi^2+a^2)+\sigma P_1]+\sigma P_1 R_1 a^2[\tau(\pi^2+a^2)+\sigma P_1][1+FP_1\sigma]-\sigma P_1 R' a^2[\pi^2+a^2+\sigma P_1]$$
$$[1+FP_1\sigma]-\sigma P_1 P_2[1+F\varepsilon P_1\sigma][\pi^2+a^2][\pi^2+a^2+P_1\sigma][\tau(\pi^2+a^2)+\sigma P_1]=0. \quad (20)$$

Moreover, equation (20) can be rewritten as

$$\rho[\pi^2+a^2][\pi^2+a^2+\sigma P_1]=\frac{R_2a^2}{\sigma P_1}[\pi^2+a^2+\sigma P_1]+R_1a^2-\frac{R'a^2[\pi^2+a^2+\sigma P_1]}{[\tau(\pi^2+a^2)+\sigma P_1]}$$
$$-\frac{P_2(\pi^2+a^2)[\pi^2+a^2+\sigma P_1][1+FP_1\sigma\varepsilon]}{[1+FP_1\sigma]}$$

$$\rho[1+X][1+X+\sigma P_1]=\frac{R_3X}{\sigma P_1}[1+X+\sigma P_1]+R_1X-\frac{R_4X[1+X+\sigma P_1]}{[\tau(1+X)+\sigma P_1]}$$
$$-\frac{P_2(1+X)[1+X+\sigma P_1][1+FP_1\sigma\varepsilon]}{[1+FP_1\sigma]} \quad (21)$$

where

$$R_3=\frac{R_2}{\pi^4},R_1=\frac{R}{\pi^4},R_4=\frac{R'}{\pi^4},X=\frac{a^2}{\pi^2},\text{ and }F_1=\frac{F}{\pi^2}.$$

As discussed earlier, the Principle of Exchange of Stabilities being not valid for the present problem, the marginal state is governed by $\square = i\square_i$, where $\square_i$ is real. Therefore, putting $\square = i\square_i$ in equation (21), the equation at the marginal state is obtained as

$$[1+X][(1+X)i\sigma_i+\sigma_i^2P_1]=-\frac{R_3X}{\sigma_iP_1}[i[1+X]-\sigma_iP_1]+R_1X$$
$$-\frac{R_4X[\tau(1+X)^2+\sigma_i^2P_1^2]-i\sigma_iP_1[1+X][1-\tau]}{[\tau^2[1+X]^2+\sigma_i^2P_1^2]}-\frac{P_2X[1+X]}{[1+F_1^2P_1^2\sigma_i^2]}$$
$$[[1+X]+[1+X]F_1^2P_1^2\sigma_i^2\varepsilon+F_1^2P_1^2\sigma_i^2(1-\varepsilon)+i(\sigma_iP_1-[1+X]F_1^2P_1^2\sigma_i^2(1-\varepsilon)+F_1^2P_1^3\sigma_i^3\varepsilon] \quad (22)$$

The real part of equation (22) is given by

$$R=\frac{\pi^2}{X}\left[-[1+X]\sigma_i^2P_1-R_3X+\frac{R_4X[\tau(1+X)^2+\sigma_i^2P_1^2]}{[\tau^2[1+X]^2++\sigma_i^2P_1^2]}\right.$$
$$+\frac{P_2[1+X]}{[1+F_1^2P_1^2\sigma_i^2]}[[1+X]+[1+X]F_1^2P_1^2\sigma_i^2\varepsilon+F_1^2P_1^2\sigma_i^2(1-\varepsilon)] \quad (23)$$

Also, the imaginary part of equation (22) is given by

$$[1+X]^2\sigma_i=\frac{R_3X[1+X]}{\sigma_iP_i}+\frac{R_4X\sigma_iP_1[1+X][1-\tau]}{[\tau^2[1+X]^2++\sigma_i^2P_1^2]}$$
$$-\frac{P_2[1+X][\sigma_iP_i-[1+X]F_1P_1\sigma_i(1-\varepsilon)+F_1^2P_1^3\sigma_i^3\varepsilon]}{[1+F_1^2P_1^2\sigma_i^2]}$$

This leads to the following sixth degree equation in σ_i :

$$A\sigma_i^6+B\sigma_i^4+C\sigma_i^2+D=0, \quad (24)$$

where,
$$A=(1+X)P_1^3F_1^2[(1+X)P_1+P_2F_1\varepsilon]$$
$$B=(1+X)^2P_1^3+(1+X)^4F_1^2P_1^3\tau^2+R_3X(1+X)F_1^2P_1^4$$
$$-R_4X[(1+X)(1-\tau)F_1^2P_1^4(1+\varepsilon)-(1+X)^2P_2F_1^3P_1^4(1-\varepsilon),$$
$$C=\tau^2(1+X)^4P_1+R_3X(1+X)^3[P_1^2\tau^2F_1^2+P_1^2+\tau^2]-R_4P_1^2X(1+X)(1-\tau)$$
$$+P_2P_1^2(1+X)[1-F_1(1+X)(1-\varepsilon)].$$

And $D = R_3X[1 + X]^3□^2$.

Equation (23) yields the Rayleigh number R at the marginal state and the frequency of oscillations $□_i$ is given by equation (24).

We now discuss the existence of overstable marginal state under various situations.

Case – 1 : The case when

(i) $R_3 > 0$ $\left(\Rightarrow \frac{df}{dz} > 0\right)$

(ii) $1 – □ < 0$ $\left(\Rightarrow k_S > k_T\right)$

And (iii) $□ > 1$

Observe that

(i) When $□ > 1$, then A is positive

and (ii) $R_3 > 0$ and $□ > 1$ ensure that both B and C are positive definite. Therefore, there is no value of $□_i$ for which σ_i^2 is positive (see equation (24)). It follows that the overstability cannot occur at the marginal state. However, the situation contrary to our assumed conditions in this case, in general, does not automatically guarantee the occurrence of overstability.

In fact, for $R_3 > 0$, $□ > 1$, marginal state may exist in the following two cases :

(i) Let □ satisfies the inequality $0 < □ < 1$. Then □ makes either B or C negative and hence marginal state may exist.

(ii) If $R_3 > 0$, $□ > 1$ and $□ > 1$, the marginal state may exist if the visco-elastic parameter F_1 is so large that it makes either of B or C negative and satisfying the inequality

$$4[BD – C^2][AC – B^2] > [AD – BC]^2.$$

Thus, we see that the visco-elasticity has an effective role in instability criteria as there is a sufficient room for the existence of marginal state even if $R_3 > 0$, $□ > 1$ and $□ > 1$.

Case – 2 : When $R_3 < 0$, one of the roots of equation (24) is always positive irrespective of the other parameters. Therefore the marginal state and overstability essentially occur.

(C) Nature of Non-Oscillating Modes :

For $R_3 > 0$, $k_S > k_T$ and $□ > 1$, the only modes that may exist are non-oscillatory modes for which $□_i = 0$ and $□ = □_r$ ($□_r$ is real). Hence substitution of $□ = □_r$ and $w = w_0 \sin □z$ in equation (13) gives

$$D_0\sigma_r^5+D_1\sigma_r^4+D_2\sigma_r^3+D_3\sigma_r^2+D_4\sigma_r+D_5=0, \tag{25}$$

where

$$D_0=P_1^2(\pi^2+a^2)F,$$
$$D_1=P_1^3(\pi^2+a^2)[-P_1F\varepsilon P_2+1+F(\pi^2+a^2)(1+\tau)],$$
$$D_2=-FP_1^3a^2(R_1-R'+R)+(\pi^2+a^2)^2P_1^2[(\pi^2+a^2)\tau-P_2F\varepsilon(1+\tau)]-P_2P_1^3(\pi^2+a^2)],$$
$$D_3=-FP_1^2a^2(\pi^2+a^2)[R(1+\tau)+R_1\tau-R']-P_1^2P_2(\pi^2+a^2)[F\varepsilon\tau(\pi^2+a^2)+(1+\tau)]$$

$$+P_1(\pi^2+a^2)^3\tau - P_1^2a^2(R+R_1-R'),$$

$$D_4 = -P_1a^2(\pi^2+a^2)[R_1\tau - R' + R(1+\tau)] - \tau P_1(\pi^2+a^2)[a^2RF + P_2(\pi^2+a^2)]$$

and $$D_5 = -Ra^2\tau(\pi^2+a^2).$$

Characteristic equation (25) is the fifth degree equation in □r with real coefficients and thus has five roots, which may be real. Since $R_3 > 0$, □ > 1 and the constant term D_5 in the characteristic equation being negative, hence has at least one positive real root and thus making the system unstable. Thus, we conclude that non-oscillatory modes are unstable in nature.

VARIATIONAL PRINCIPLE

A variational principle can be established for the present problem following Chandrashekhar **[3]**.
Let one of the characteristic values be n_i and let the corresponding solutions be denoted by a subscript "i", then

$$-k^2L = \rho_0\left[n + \frac{\nu(1+\lambda_0 n)}{k_1(1+\lambda n)}\right]Dw \tag{26}$$

and $$-DL = \rho_0\left[nw - \frac{g}{n}\left(\frac{df}{dz}\right)w - g\alpha\theta + g\alpha'\Gamma\right] + \frac{\nu}{k_1}\rho_0\frac{[1+\lambda_0 n]}{[1+\lambda n]}w, \tag{27}$$

where L are the forms of F(z) in □p
From equation (26), we have,

$$-DL_i = \rho_0\left[n_iw_i - \frac{g}{n_i}\left(\frac{df}{dz}\right)w_i - g\alpha\theta_i + g\alpha'\Gamma_i\right] + \frac{\nu}{k_1}\rho_0\frac{[1+\lambda_0 n_i]}{[1+\lambda n_i]}w_i, \tag{28}$$

Also from equation (27)

$$-k^2L_i = \rho_0\left[n + \frac{\nu(1+\lambda_0 n)}{k_1(1+\lambda n)}\right]Dw_i \tag{29}$$

Let n_j be a characteristic value different from n_i, and let subscript 'j' distinguishes the corresponding solutions. We multiply equations (28) and (29) respectively by w_j and DW_j and integrate them with respect to z from z = 0 and z = d using the boundary conditions

$$\left.\begin{array}{l} w = D^2w = X = Dz = 0 \\ \theta = \Gamma = 0 \end{array}\right] \text{at } z = 0 \text{ and } z = d. \tag{30}$$

We have

$$\int_0^d (DL_i)w_j dz = \int_0^d\left[n_i + \frac{g}{n_i}\left(\frac{df}{dz}\right) - \frac{\nu}{k_1}\frac{(1+\lambda_0 n_i)}{(1+\lambda n_i)}\right]w_iw_jdz$$
$$+\rho_0\int_0^d g\alpha\theta_i w_j dz - \int_0^d g\alpha'\rho_0\Gamma_i w_j dz \tag{31}$$

and $$-\int_0^d k^2L_i(Dw_j)dz = \int_0^d\left[n_i + \frac{\nu}{k_i}\frac{(1+\lambda_0 n_i)}{(1+\lambda n_i)}\right]Dw_iDw_jdz. \tag{32}$$

Substituting the characteristic values n_j, w_j and □j and multiplying them by □i and □i respectively, integrating the same from z = 0 and z = d under the boundary conditions (30) and substituting the results in equation (31), we get

$$\int_0^d L_i(Dw_j)dz = \int_0^d \rho_0\left[n_i + \frac{g}{n_i}\left(\frac{df}{dz}\right) - \frac{\nu}{k_1}\frac{(1+\lambda_0 n_i)}{(1+\lambda n_i)}\right]w_iw_jdz$$

$$+\int_0^d \rho_0 \frac{g\alpha}{\beta}[k_T D\theta_i D\theta_j + k_T k^2 \theta_i \theta_j + n_j \theta_i \theta_j]dz$$

$$+\int_0^d \rho_0 \frac{g\alpha'}{\beta}[k_S D\Gamma_i D\Gamma_j + k_S k^2 \Gamma_i \Gamma_j + n_j \Gamma_i \Gamma_j]dz \tag{33}$$

Also integrating the L.H.S. of equation (32) by parts and using the boundary conditions (30), we get

$$-\int_0^d k^2 L_i (Dw_j)\, dz = \int_0^d -\rho_0 \left[n_i + \frac{\nu}{k_1}\frac{(1+\lambda_0 n_i)}{(1+\lambda n_i)} \right] Dw_i Dw_j dz. \tag{34}$$

Putting i = j and suppressing the subscript, equations (33) and (34) yield

$$-n\int_0^d \left[w^2 + \frac{1}{k^2}(Dw)^2 \right] dz - \left[\frac{(1+\lambda_0 n)}{(1+\lambda n)} \right] \int_0^d \frac{\nu}{k_1}(w^2 + \frac{1}{k^2}(Dw)^2 dz$$

$$+\frac{g}{n}\int_0^d \left(\frac{df}{dz} \right)^2 w^2 dz + n\int_0^d \frac{g\alpha\theta^2}{\beta} dz + \int_0^d \frac{g\alpha k_T}{\beta}[(D\theta)^2 + k^2(\theta)^2]dz$$

$$-\int_0^d \frac{g\alpha'\Gamma'}{\beta'} dz - \int_0^d \frac{g\alpha' k_S}{\beta'}[(D\Gamma)^2 + k^2(\Gamma)^2]dz = 0 \tag{35}$$

Equation (35) provides a basis for the variational formulation of the problem as discussed below:
Let

$$\left. \begin{aligned} J_1 &= \int_0^d \left[w^2 + \frac{1}{k^2}(Dw)^2 \right] dz, J_2 = \int_0^d \frac{\nu}{k_1}\left[w^2 + \frac{1}{k^2}(Dw)^2 \right] dz, \\ J_3 &= \int_0^d \left(\frac{df}{dz} \right) w^2\, dz, J_4 = \int_0^d \frac{g\alpha\theta^2}{\beta} dz, \\ J_5 &= \int_0^d \frac{g\alpha'\Gamma^2}{\beta'} dz, J_6 = \int_0^d \frac{g\alpha k_T}{\beta}[(D\theta)^2 + k^2(\theta)^2]dz, \\ \text{and } J_7 &= \int_0^d \frac{g\alpha' k_S}{\beta'}[(D\Gamma)^2 + k^2(\Gamma)^2]dz, \end{aligned} \right] \tag{36}$$

With the help of equation (36), equation (35) can be written as

$$-nJ_1 - \left[\frac{(1+\lambda_0 n)}{(1+\lambda n)} \right] J_2 + J_3 \frac{g}{n} + (J_4 - J_5)n + J_6 - J_7 = 0$$

$$-n[J_1 - J_4 + J_5] + \frac{g}{n} J_3 - \left(\frac{1+\lambda_0 n}{1+\lambda n} \right) J_2 + J_6 - J_7 = 0 \tag{37}$$

Let us now consider the variations δn in n caused by the first order small variations δw, $\delta\theta$ and $\delta\Gamma$ in w, θ and Γ respectively.
Further, we assume that δw, $\delta\theta$ and $\delta\Gamma$ satisfy the boundary conditions (30). The changes in w, θ and Γ lead to the corresponding changes in $J_i^{'}s$, denoted by $\delta J_i^{'}s$. We can analyse these changes with the help of equation which gives

$$-\delta n\left[J_1-J_4+J_5+\frac{g}{n^2}J_3-\frac{(\lambda_0-\lambda)}{(1+\lambda n)^2}J_2\right]-\left[\delta J_1-\delta J_4+\delta J_5-\frac{g}{n}\delta J_3+\left(\frac{1+\lambda_0 n}{1+\lambda n}\right)\delta J_2\right]$$
$$-\delta J_6-\delta J_7=0 \quad (38)$$

We, now use the expressions for $J_i^{'}s$ given by (36) to evaluate $\delta J_i^{'}s$. Integrating by parts a suitable number of times and using (30), we find

$$\left.\begin{aligned}
&\frac{1}{2}\delta J_1=\int_0^d \frac{1}{k^2}(D^2-w^2)w dz, \frac{1}{2}\delta J_2=\int_0^d \frac{\nu}{k_1 k^2}(D^2-k^2)\, w\, dz,\\
&\frac{1}{2}\delta J_3=\int_0^d \left(\frac{df}{dz}\right)(dw).w dz, \frac{1}{2}\delta J_4=\int_0^d \frac{g\alpha}{\beta}(\delta\theta)\theta dz,\\
&\frac{1}{2}\delta J_5=\int_0^d \frac{g\alpha'}{\beta'}(\delta\Gamma)\Gamma dz, \frac{1}{2}\delta J_6=-\int_0^d \frac{g\alpha k_T}{\beta}\delta\theta(D^2-k^2)\theta dz,\\
\text{and }&\frac{1}{2}\delta J_7=\int_0^d \frac{g\alpha' k_S}{\beta'}\delta\Gamma(D^2-k^2)\theta dz.
\end{aligned}\right] \quad (39)$$

Combining equations (38) and (39) and using equations (30) to (37) in it and rearranging the terms, we get

$$-\frac{\delta n}{2}\left[J_1-J_4+J_6+\frac{g}{n^2}J_3+\frac{\lambda_0-\lambda}{(1+\lambda n)^2}J_2\right]+\int_0^d g\alpha(w\delta\theta-\theta\delta\, w)\, dz$$
$$+\int_0^d g\alpha'(w\delta\Gamma-\Gamma\delta\, w)\, dz=0 \quad (40)$$

Multiplying equation of magnetic field by $\delta\theta$ and integrating w.r.t. z from z = 0 to z = d, we get

$$\int_0^d n\theta\delta\theta dz+\int_0^d [k_T k^2\theta\delta\theta-k_T(D^2\theta)\delta\theta]dz=\int_0^d \beta w\delta\theta dz. \quad (41)$$

On taking first order variations in equation of magnetic field, multiplying it by □ and integrating w.r.t. z from z = 0 and z = d, we get

$$\int_0^d \delta n\theta^2 dz+\int_0^d n\theta\delta\theta dz+\int_0^d [k_T k^2\theta\delta\theta-k_T\theta D^2\delta\theta]dz=\int_0^d \beta\theta\delta w dz. \quad (42)$$

Subtracting equation (42) from equation (41) and integrating by parts, using boundary conditions (30), we get after some rearrangement of the terms

$$g\alpha\int_0^d [w\delta\theta-\theta\delta w]dz=-\delta n J_4 \quad (43)$$

Proceeding similarly, we get

$$g\alpha'\int_0^d [w\delta\Gamma-\Gamma\delta w]dz=-\delta n J_5 \quad (44)$$

Using equations (43) and (44) equation (40) reduces to

$$-\frac{\delta n}{2}\left[J_1-J_4+J_6+\frac{g}{n^2}J_3+\frac{(\lambda_0-\lambda)}{(1+\lambda n)^2}J_2\right]=0. \quad (45)$$

Now, it is evident from equation (37) that the quantity within [ ] on the L.H.S. of equation (45) can not vanish. Hence, □n = 0.
Therefore, equation (35) provides a basis for the variational formulation of the problem under investigation.

NUMERICAL COMPUTATIONS

The effect of various parameters on the instability criteria is studied with the help of numerical computations using variational principle.
Let us consider the trial solution for w, □ and □ corresponding to the lowest mode as $w = w_0 \sin lz$, □ = $□_0$ sin lz and □ = $□_0$ sin lz, where w_0, $□_0$ and $□_0$ are constants and l = □/d. Also, let f(z) = 1 + □z, where □ is a constant.
The substitution of the trial solution in equation (35) and its further simplification ultimately gives, in dimensionless form, the fifth degree equation in □, namely,

$$A_0\sigma^5 + A_1\sigma^4 + A_2\sigma^3 + A_3\sigma^2 + A_4\sigma + A_5 = 0, \qquad (46)$$

where

$$A_0 = P_1^3 F(1+y^2),$$

$$A_1 = P_1^2(1+y^2)[1+P_1P_2\,\varepsilon F(1+y^2)(1+\tau)] + P_1^2(1+y^2)P_2 + P_1^2 F y^2(R' - R - R_5\, p_1\, b\pi)],$$

$$A_3 = \pi^4(1+y^2)^3 z[1+\varepsilon P_1P_2F + P_1Fy^2\pi^2(1+\pi^2)[R' - \tau R - Rb\pi\, p_1(1+\pi)] + \pi^2 P_1P_2(1+y^2)^2(1+\tau) + P_1y^2(R' - R - R_5 b\pi\, p_1)$$

$$A_4 = \pi^4\tau(1+y^2)^3P_2 - Rb\pi^3 y^2P_1(1+y^2)(\pi^2\tau F(1+y^2) + (1+\tau)] + \pi^2 y^2(1+\pi^2)(R' - R\tau)$$

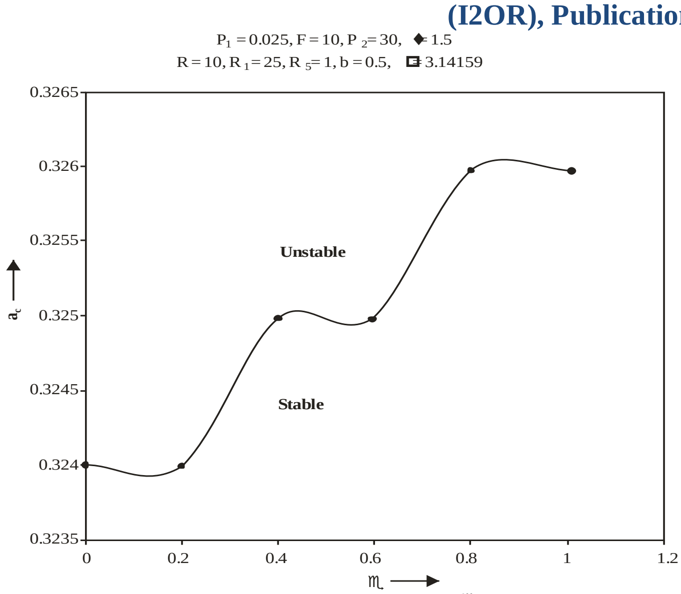
and $A_5 = -Rb\pi^5 y^2 \tau(1+y^2)^2.$

where the quantities have been non-dimensionalized as

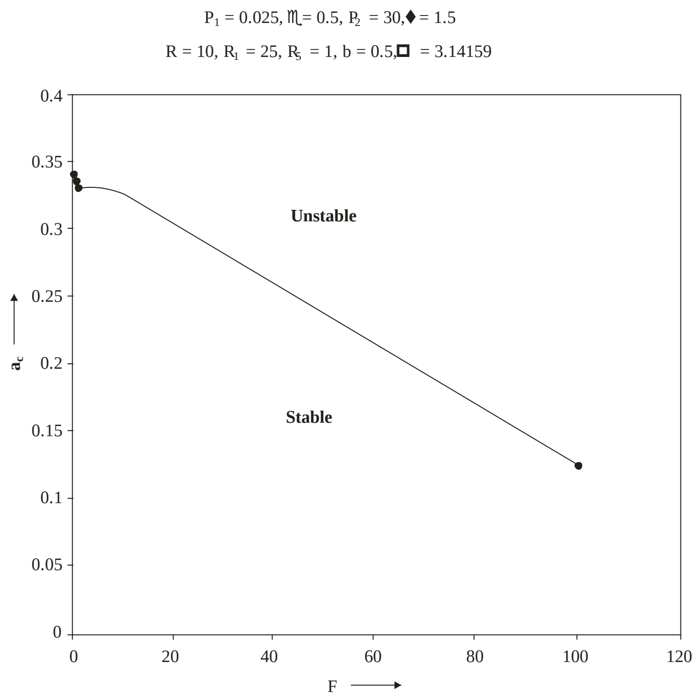
$$y = \frac{k}{l}, \quad b = \frac{\psi}{l}, \quad \sigma = \frac{nd^2}{\nu}, \quad P_1 = \frac{\nu}{k}, \quad \tau = \frac{k_S}{k_T},$$

$$P_2 = \frac{d^2}{k_1}, \quad R = \frac{g\alpha\beta d^4}{k_T\nu}, \quad R' = \frac{g\alpha'\beta' d^4}{k_T\nu} \quad \text{and} \quad R_5 = \frac{gd^3}{\nu^2}.$$

The roots of equation (46) have been located for different values of b, y, R, R´ and P_2 by making use of numerical computations. The results are illustrated graphically in Figs. 1 and 2. From Fig. 1 and Fig. 2, we find that □ has a stabilizing effect on the system.



Critical Wave Number a_c Vs. ♏
Fig. 1



Critical Wave Number a_c Vs. F
Fig. 2

CONCLUSION

An analysis of the problem and the discussions of the results lead to the conclusion that the Principle of Exchange of Stability is not valid for this problem and the frequency of oscillations and the Rayleigh number in the marginal state are given by equation (23) and equation (24).

Further, we find for density distribution with positive gradient and for $k_s > k_T$, the overstable marginal state does not exist and we have only non-oscillatory modes which make the system unstable. For density distribution with negative gradient, the marginal state and overstabSe solution exist, irrespective of the values of other parameters.

REFERENCES

[1] **Benard, H.** : Revue generale des sciences pures et apliques, **11**: 1261-71 and 1309-28 (1900).
[2] **Rayleigh, L.** : Phil. Mag. 32 : 529 (1926).
[3] **Chandrashekhar, S.** : Hydrodynamic and Hydromagnetic Stability, Oxford University Press, London (1961).
[4] **Stomell, H., Arones, A. B. and Bhanchord, D.** : Deep Sea Res. **3** : 152 (1956).
[5] **Stern, M.E.** : Tellus **12** : 172 (1960).
[6] **Veronis, G.** : J. Fluid Mech. **23** : 1(1965).
[7] **Nield, D.A.** : J. Fluid Mech. **29** : 545 (1967).
[8] **Toms, B.A. and Strabridge, D. J.** : Trans. Faraday Soc., **49** : 1205 (1953).
[9] **Oldroyd, J. G.** : Proc. Roy. Soc. (London), **A245**, 278-297 (1958).
[10] **Sharma, R. C.** : J. Maths. Phys. Sci. **12** : 603 (1978).
[11] **Khare, H. C. and Sahai, A. K.** : Proc. Nat. Acad. Sci. India **62(A), IV**, 673-689(1992).